A renormalisation group approach for two-dimensional percolating systems: Honeycomb bond lattice and Kagome lattice

1979 J. Phys. A: Math. Gen. 12 1541

(http://iopscience.iop.org/0305-4470/12/9/021)

J. Phys. A: Math. Gen., Vol. 12, No. 9, 1979. Printed in Great Britain

A renormalisation group approach for two-dimensional percolating systems: Honeycomb bond lattice and Kagomé lattice

C Murase† and Y Yuge‡

† Laboratory of Physics, Faculty of General Education, Tokyo University of Agriculture and Technology, Fuchu-shi, Tokyo, Japan.

‡ Institute of Agricultural Engineering, University of Tsukuba, Sakura-mura, Niihari-gun, Ibaraki-ken, Japan.

Received 7 July 1978, in final form 4 December 1978

Abstract. A renormalisation group approach to two-dimensional percolation problems on the honeycomb lattice (bond) and the Kagomé lattice (site and bond) is developed using a scaling transformation in real space. A finite cluster approach, which we call the electrode method, gives the location of the fixed point p^*, the eigenvalue λ and the correlation length critical exponent ν; the results are $p^* = 0{\cdot}6308$, $\lambda = 1{\cdot}669$ and $\nu = 1{\cdot}353$ for both the honeycomb lattice (bond) and the Kagomé lattice (site), and $p^* = 0{\cdot}4697$, $\lambda = 1{\cdot}577$ and $\nu = 1{\cdot}522$ for the Kagomé lattice (bond). The fixed point p^* for the honeycomb lattice (bond) and the Kagomé lattice (site) is in good agreement with the exact critical percolation probability obtained by Sykes and Essam.

1. Introduction

In previous papers we have presented a simple method, which we call the 'electrode method', of a renormalisation group approach in real space (Yuge and Murase 1978, Yuge 1978). In these papers we studied the site percolation problems on the square lattice (Yuge and Murase 1978) and on the triangular lattice (Yuge 1978), and also the bond percolation problem on the square lattice (Yuge 1978). In this paper we apply this electrode method to the percolation problems on the two-dimensional honeycomb lattice (bond) and Kagomé lattice (bond and site), and calculate the critical percolation probability p_c, eigenvalue λ_1 and the correlation length critical exponent ν.

The basic method of this approach can be set up in terms of Kadanoff's original picture, the block spin picture (Kadanoff 1966), using the scaling transformation in real space. The method of Kadanoff's picture has been formulated recently and applied to the study of critical behaviour in percolating systems (Harris *et al* 1975, Young and Stinchcombe 1975, Stinchcombe and Watson 1976). If the percolation probability p on an original lattice scales into a new percolation probability p' on a new lattice by the renormalisation transformation R,

$$p' = R(p), \tag{1}$$

then the fixed point p^* is determined by the relation

$$p^* = R(p^*). \tag{2}$$

The non-trivial fixed point also gives an approximation for the critical percolation probability p_c. The linearised form of the renormalisation transformation near the fixed point has the eigenvalues λ_i with $\lambda_1 > 1 > \lambda_2 \ldots$. The correlation length critical exponent ν is related to the maximum eigenvalue λ_1 by

$$\nu = \ln b / \ln \lambda_1 \qquad \lambda_1 = \mathrm{d}R(p)/\mathrm{d}p|_{p=p^*} \tag{3}$$

where b is the change of scale of length.

2. Honeycomb lattice (bond)

Our scaling procedure is defined by a renormalisation transformation on finite lattices sandwiched by two electrodes. An illustration of the basic scaling procedure on this lattice is provided in figures 1(a)–(c). The cluster of bonds on the original honeycomb lattice (full line) enclosed by the broken line in figures 1(a) and (b) scales into a new lattice (full line) in figure 1(c) with a scale factor $b = 2$. As shown in figure 1(c), the renormalised probability p' of a bond on the new lattice is determined as the probability of the cluster being conductive when the cluster is sandwiched between two plane electrodes AB and CD made of perfect conductor. There are three equivalent transformations as shown in figure 1(a) and two transformations shown in figure 1(b).

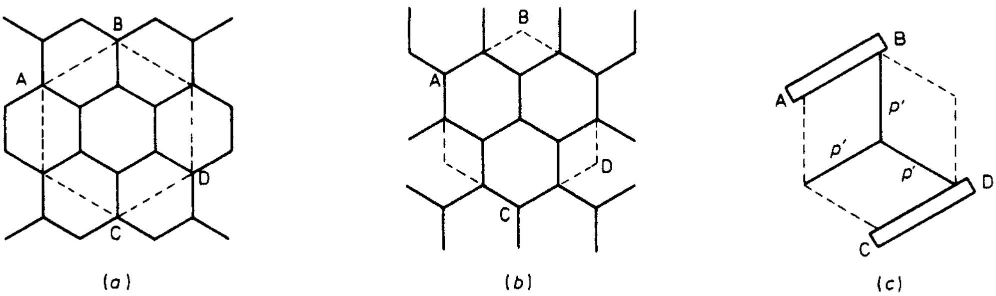

Figure 1. Renormalisation transformation on the honeycomb bond lattice. In (a) and (b) the full line is an original lattice; the cluster of bonds enclosed by the broken line ABCD is transformed into a new lattice ABCD shown in (c) with the scale factor $b = 2$. (c) Combination of paths for the new probability p'. New bonds (full line) are sandwiched between two plane electrodes AB and CD.

The three transformations can be generated from each other by the spatial translation on the original lattice to a bond direction by a bond length. We denote the probability for the two electrodes being conductive by $T_1(p)$ and $T_2(p)(= T_3(p))$ corresponding to the transformation in figure 1(a) and to the two transformations in figure 1(b), respectively. Then the total average probability $T(p)$ for which two electrodes AB and CD become conductive can be given by the arithmetic average of the three probabilities:

$$T(p) = \tfrac{1}{3}(T_1(p) + T_2(p) + T_3(p)). \tag{4}$$

On the other hand, the probability for two electrodes being conductive on the new lattice is p'^2. Then the renormalisation transformation by the electrode method can be given by

$$R(p) = (T(p))^{1/2}. \tag{3}$$

We can obtain the probabilities $T_i(p)(i = 1, 2, 3)$ from the paths which contribute to the conductance of the clusters according to exclusion–inclusion principle:

$$T_1(p) = 2p^4 + 4p^5 - 2p^6 - 4p^7 - 7p^8 + 12p^9 - 4p^{10} \quad (6)$$

$$T_2(p) = T_3(p)$$
$$= 2p^4 + 2p^5 + p^6 - 4p^7 - 2p^8 + 4p^9 - 7p^{10} + 2p^{11} + 7p^{12} - 4p^{13}. \quad (7)$$

From the renormalisation transformation (5), the fixed point p^*, the maximum eigenvalue λ_1 and the correlation length critical exponent ν can be calculated and are given by

$$p^* = 0{\cdot}6308 \qquad \lambda_1 = 1{\cdot}669 \qquad \nu = 1{\cdot}353. \quad (8)$$

3. Kagomé lattice (site)

In general, any bond problem can be transformed into an equivalent site problem on a different graph called the covering graph (Fisher and Essam 1961). Using this bond-to-site transformation we can show that the bond problems on the honeycomb lattice are transformed into the site problems on the Kagomé lattice and, correspondingly, any renormalisation group transformation on the honeycomb lattice can be transformed into an equivalent renormalisation group transformation on the Kagomé lattice. Using this property, we can find out the equivalent renormalisation transformations on the Kagomé lattice in this case of the electrode method. In figures 2(*a*)–(*c*) we show the corresponding procedure for the renormalisation group transformation on the Kagomé lattice. The cluster of sites (full circles) enclosed by the broken line in figures 2(*a*) and (*b*) is sandwiched between two plane electrodes AB and CD. The cluster of sites scales into a new Kagomé lattice in figure 2(*c*). The renormalisation transformation of the Kagomé lattice can be defined by the same procedure mentioned in the case of the honeycomb lattice, i.e. by the probability for the two electrodes being conductive. Therefore the renormalisation transformation $R(p)$ of the Kagomé lattice becomes

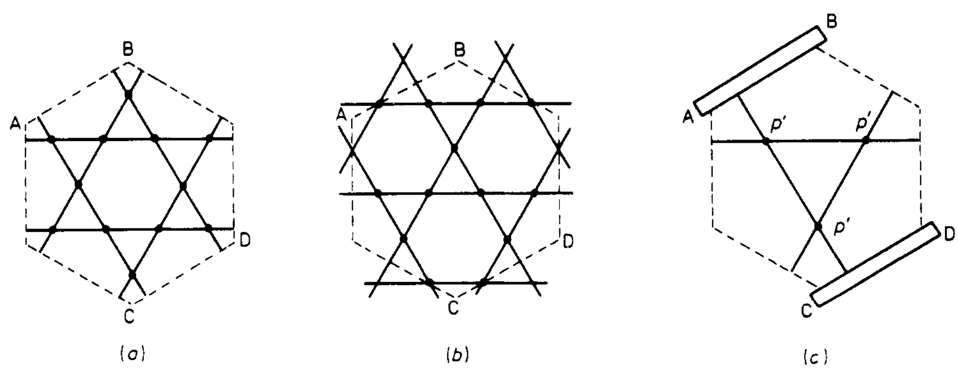

Figure 2. Renormalisation transformation on the Kagomé site lattice. In (*a*) and (*b*) the full line is an original lattice; the cluster of sites enclosed by the broken line ABDC is transformed into a new lattice ABDC shown in (*c*) with the scale factor $b = 2$. (*c*) Combination of paths for the new probability p' are sandwiched between two plane electrodes AB and CD.

exactly same as that of the honeycomb lattice, and is given by equations (5)–(7). Then the fixed point p^*, the maximum eigenvalue λ_1 and the correlation length critical exponent ν of the site problems on the Kagomé lattice are also given by equation (8).

4. Kagomé lattice (bond)

In this section we consider the bond problems on the Kagomé lattice. An illustration of the basic scaling procedure on this lattice is provided in figures 3(a) and (b). The cluster of bonds, which is enclosed by the broken line in figure 3(a), on the original Kagomé lattice (full line) scales into a new lattice (full line) in figure 3(b) with a scale factor $b = 2$.

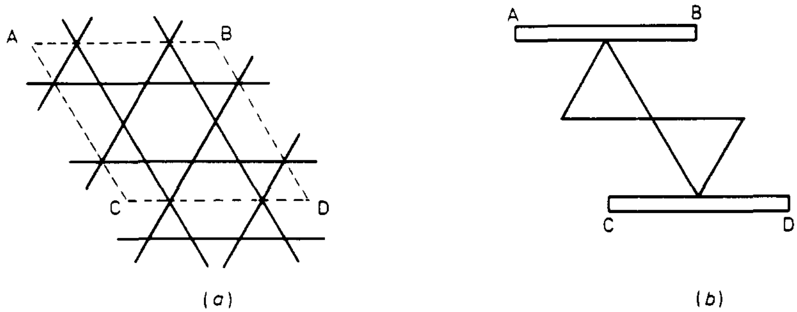

Figure 3. Renormalisation transformation on the Kagomé bond lattice. In (a) the full line is an original lattice; the cluster of bonds enclosed by the broken line ABDC is transformed into a new lattice shown in (b) with the scale factor $b = 2$. AB and CD are two plane electrodes.

AB and CD in figures 3(a) and (b) are two plane electrodes. If we denote the probability for two electrodes being conductive on the original lattice by $T_k(p)$, we can calculate this exactly:

$$T_k(p) = 4p^4 + 16p^5 + 6p^6 - 40p^7 - 54p^8 - 74p^9 + 89p^{10} + \ldots. \tag{9}$$

All coefficients of each power p^n ($n = 0, 1, \ldots, 24$) of $T_k(p)$ are listed in table 1. On the other hand, if we denote the probability for two electrodes being conductive on the new lattice by $T'_k(p')$, this becomes

$$T'_k(p') = (p' + p'^2 - p'^3)^2. \tag{10}$$

The scaled probability p' is determined so that the two probabilities $T_k(p)$ and $T'_k(p')$ become equal:

$$T'_k(p') = T_k(p). \tag{11}$$

This relation determines the renormalisation transformation $R(p)$ of equation (1) on the Kagomé lattice (bond). Using relations (2) and (3), we can calculate the fixed point, the maximum eigenvalue λ_1 and the correlation length critical exponent ν of the bond problems on the Kagomé lattice. The results are given by

$$p^* = 0{\cdot}4697 \qquad \lambda_1 = 1{\cdot}577 \qquad \nu = 1{\cdot}522. \tag{12}$$

Table 1. Coefficients of p^n of the probability $T_k(p)$ on the Kagomé lattice (bond).

n of p^n	Coefficient	n of p^n	Coefficient
0	0	13	−2276
1	0	14	1006
2	0	15	4012
3	0	16	−3723
4	4	17	−2808
5	16	18	5658
6	6	19	−1708
7	−40	20	−2373
8	−54	21	2654
9	−74	22	−1199
10	89	23	270
11	706	24	−25
12	−140		

5. Discussion

Our results of the fixed point $p^* = 0{\cdot}6308$ for the honeycomb lattice (bond) and Kagomé lattice (site) show excellent agreement with the exact critical percolation probability $p_c = 0{\cdot}6527$ (Sykes and Essam 1964). The correlation length critical exponent $\nu = 1{\cdot}353$ also shows close agreement with the value $\nu = 1{\cdot}34 \pm 0{\cdot}02$ obtained by series expansions for bond percolation on the triangular lattice (Dunn *et al* 1975). These results on the honeycomb lattice (bond) and the Kagomé lattice (site) show that our renormalisation procedure by the electrode method (equations (4)–(7)) gives a very good renormalisation transformation in real space. Since our method is very simple, the electrode method can be applied to the study of other critical behaviour of percolation problems on these lattices.

Our result $p^* = 0{\cdot}4697$ for the Kagomé lattice (bond) is in good agreement with the result $0{\cdot}435 \pm 0{\cdot}036$ of the Monte Carlo experiment obtained by Dean (1963). The critical exponent $\nu = 1{\cdot}522$ for the Kagomé lattice (bond), however, is different from the estimated value $\nu = 1{\cdot}35 \pm 0{\cdot}02$ from the scaling hypothesis (Gaunt and Sykes 1976). This difference will be due to the scale of the renormalisation. For the case of the Kagomé lattice (bond), the scaling factor is only $b = 2$. If we can calculate the cases for $b = 3, 4, \ldots$, the correlation length critical exponent will become smaller as in the case of other lattices, for example the square lattice (site) and the triangular lattice (site) (Yuge and Murase 1978, Yuge 1978). Unfortunately even the case with $b = 3$ on the Kagomé lattice (bond) includes a large number of bonds (54) and it is difficult to calculate the probability $T_k(p)$ of this lattice exactly.

Our renormalisation procedure (electrode method) in real space has one characteristic feature: its symmetry. Our renormalisation procedures are chosen so that the new renormalised lattice maintains the symmetry of the original lattice. For example, the arithmetic average of three probabilities (equation (4)) corresponding to three renormalisation procedures guarantees the symmetry on the new lattice. We think this symmetry-conserving property of our electrode method gives good results for three percolating systems; we also think this property is a very important condition in the study of critical properties of percolating systems using the renormalisation procedure in real space.

References

Dean P 1963 *Proc. Camb. Phil. Soc.* **59** 397–410
Dunn A G, Essam J W and Ritchie D S 1975 *J. Phys. C: Solid St. Phys.* **8** 4219–35
Fisher M F and Essam J W 1961 *J. Math. Phys.* **2** 609–19
Gaunt D S and Sykes M F 1976 *J. Phys. A: Math. Gen.* **9** 1109–16
Harris A B, Lubensky T C, Holcomb W K and Dasgupta C 1975 *Phys. Rev. Lett.* **35** 327–30
Kadanoff L P 1966 *Physica* **2** 263
Stinchcombe R B and Watson B P 1976 *J. Phys. C: Solid St. Phys.* **9** 3221–47
Sykes M F and Essam J W 1964 *J. Math. Phys.* **5** 117–27
Young A P and Stinchcombe R B 1975 *J. Phys. C: Solid St. Phys.* **8** L535–40
Yuge Y 1978 *Phys. Rev.* B **18** 1514–7
Yuge Y and Murase C 1978 *J. Phys. A: Math. Gen.* **11** L83–5